Solution to Assignment 1, MMAT5520

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Exercise 1.1:
1(a) $y' + y = 4e^{3x}$.
2(c) $(x^2+4)y' + 3xy = 3x;\ y(0) = 3$.
Soution:
1(a) Multiplying e^x on both sides of the equation

$$\begin{aligned} e^x \frac{dy}{dx} + e^x y &= 4e^{4x}, \\ \frac{d}{dx}(e^x y) &= 4e^{4x}, \\ e^x y &= 4\int e^{4x} dx, \\ e^x y &= e^{4x} + C, \\ y &= e^{3x} + Ce^{-x}. \end{aligned}$$

2(c) Dividing both sides by x^2+4, the equation becomes

$$\frac{dy}{dx} + \frac{3x}{x^2+4} y = \frac{3x}{x^2+4}.$$

Now, we multiply both sides by $(x^2+4)^{\frac{3}{2}}$ and get

$$\begin{aligned} (x^2+4)^{\frac{3}{2}} \frac{dy}{dx} + 3x(x^2+4)^{\frac{1}{2}} y &= 3x(x^2+4)^{\frac{1}{2}}, \\ \frac{d}{dx}((x^2+4)^{\frac{3}{2}} y) &= 3x(x^2+4)^{\frac{1}{2}}, \\ (x^2+4)^{\frac{3}{2}} y &= \int 3x(x^2+4)^{\frac{1}{2}} dx, \\ (x^2+4)^{\frac{3}{2}} y &= (x^2+4)^{\frac{3}{2}} + C, \\ y &= 1 + C(x^2+4)^{-\frac{3}{2}}. \end{aligned}$$

Since $y(0) = 3, C = 16$. Thus

$$y = 1 + 16(x^2+4)^{-\frac{3}{2}}.$$

Exercise 1.2:
2(a) $xy' - y = 2x^2 y;\ y(1) = 1$.
Soution:

$$\begin{aligned} y' &= (2x + x^{-1})y, \\ \frac{dy}{y} &= (2x + x^{-1})dx, \\ \int \frac{dy}{y} &= \int (2x + x^{-1})dx, \\ \ln y &= x^2 + \ln x + C', \\ y &= Cxe^{x^2}, \quad C = e^{C'}. \end{aligned}$$

Since $y(1) = 1, C = e^{-1}$. Thus

$$y = xe^{x^2-1}.$$

Exercise 1.3: Find the value of k so that the equation is exact and solve it:
2(c) $(2xy^2 + 3x^2)dx + (2x^k y + 4y^3)dy = 0$.
Soution: The equation is exact provided

$$\begin{aligned} \frac{\partial}{\partial y}(2xy^2 + 3x^2) &= \frac{\partial}{\partial x}(2x^k y + 4y^3), \\ 4xy &= 2kyx^{k-1}, \\ k &= 2. \end{aligned}$$

Set

$$F(x, y) = \int (2xy^2 + 3x^2)dx = x^2y^2 + x^3 + g(y).$$

We want

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= 2x^2y + 4y^3, \\ 2x^2y + g'(y) &= 2x^2y + 4y^3, \\ g'(y) &= 4y^3. \end{aligned}$$

Therefore we may choose $g(y) = y^4$ and the solution is

$$x^2y^2 + x^3 + y^4 = 0.$$

Exercise 1.4:
1(e) $x^2y' = xy + y^2$.
Soution: Rewriting the equation as

$$\frac{dy}{dx} = \frac{y}{x} + (\frac{y}{x})^2.$$

Let $u = \frac{y}{x}$, we have

$$\begin{aligned}
u + x\frac{du}{dx} &= u + u^2, \\
x\frac{du}{dx} &= u^2, \\
\frac{du}{u^2} &= \frac{dx}{x}, \\
\int u^{-2}du &= \int x^{-1}dx, \\
-u^{-1} &= \ln|x| + C, \\
-\frac{x}{y} &= \ln|x| + C, \\
y &= -\frac{x}{\ln|x| + C} \quad \text{or} \quad y = 0.
\end{aligned}$$

Exercise 1.5:
1(c) $xy' = y(x^2y - 1)$.
Soution: Let $u = y^{1-2} = y^{-1}$, then

$$\begin{aligned}
\frac{du}{dx} = -y^{-2}\frac{dy}{dx} &= -y^{-2} \cdot \frac{y(x^2y-1)}{x}, \\
\frac{du}{dx} = -x + x^{-1}y^{-1} &= -x + \frac{u}{x}, \\
\frac{du}{dx} - \frac{u}{x} &= -x.
\end{aligned}$$

Multiplying both sides by x^{-1} gives

$$\begin{aligned}
x^{-1}\frac{du}{dx} - x^{-2}u &= -1, \\
\frac{d}{dx}(x^{-1}u) &= -1, \\
x^{-1}u &= -x + C, \\
u &= -x^2 + Cx.
\end{aligned}$$

Hence

$$y = \frac{1}{Cx - x^2} \quad \text{or} \quad y = 0.$$

Exercise 1.6: Solve the differential equation by using the given substitution.
1(b) $y' = \sqrt{x+y}$; $u = x + y$.

Soution: Let $u = x + y$, then

$$\begin{aligned} \frac{du}{dx} &= 1 + \frac{dy}{dx}, \\ \frac{du}{dx} &= 1 + \sqrt{u}, \\ \frac{du}{1+\sqrt{u}} &= dx, \\ \int \frac{du}{1+\sqrt{u}} &= \int dx, \\ 2\sqrt{u} - 2\ln(1+\sqrt{u}) &= x + C, \\ 2\sqrt{x+y} - 2\ln(1+\sqrt{x+y}) &= x + C. \end{aligned}$$

Exercise 1.7:
1(a) $yy'' + (y')^2 = 0$.
2(b) $y' = \dfrac{x^2+2y}{x}$.
2(d) $xy' + 2y = 6x^2\sqrt{y}$.
Soution:
1(a) The equation reads

$$\begin{aligned} \frac{d}{dx}(yy') &= 0, \\ yy' &= C_1, \\ ydy &= C_1 dx, \\ \int ydy &= \int C_1 dx, \\ \frac{1}{2}y^2 &= C_1 x + C_2. \end{aligned}$$

2(b) Rewriting the equation as

$$y' - 2x^{-1}y = x.$$

Multiplying both sides by x^{-2} gives

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = x^{-1},$$

$$\begin{aligned} \frac{d}{dx}(x^{-2}y) &= x^{-1}, \\ x^{-2}y &= \ln|x| + C, \\ y &= x^2\ln|x| + Cx^2. \end{aligned}$$

2(d) Let $u = \sqrt{y}$, then we have

$$\frac{du}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{6x^2u - 2u^2}{2ux} = 3x - x^{-1}u.$$

$$\frac{du}{dx} + x^{-1}u = 3x.$$

Multiplying both sides by x leads

$$\begin{aligned} x\frac{du}{dx} + u &= 3x^2, \\ \frac{d}{dx}(xu) &= 3x^2, \\ xu &= x^3 + C, \\ u &= x^2 + Cx^{-1}. \end{aligned}$$

Therefore we have

$$y = (x^2 + Cx^{-1})^2 \quad \text{or} \quad y = 0.$$